

Particles, fluids and vortices

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Abstract

Classical particle mechanics on curved spaces is related to the flow of ideal fluids, by a dual interpretation of the Hamilton-Jacobi equation. As in second quantization, the procedure relates the description of a system with a finite number of degrees of freedom to one with infinitely many degrees of freedom. In some two-dimensional fluid mechanics models a duality transformation between the velocity potential and the stream function can be performed relating sources and sinks in one model to vortices in the other. The particle mechanics counterpart of the dual theory is reconstructed. In the quantum theory the strength of sources and sinks, as well as vorticity are quantized; for the duality between theories to be preserved these quantization conditions must be related.

1 Particles

The free motion of a classical particle with unit mass, moving in a smooth space with metric $g_{ij}(x)$ is described by the Lagrangean

$$L = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j, \quad (1)$$

where as usual the overdot represents a time-derivative. The Euler-Lagrange equations imply that the particle moves on a geodesic:

$$\frac{D^2 x^i}{Dt^2} = \ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0. \quad (2)$$

The canonical formulation of this theory is constructed in terms of the momenta

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij} \dot{x}^j, \quad (3)$$

and the hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j. \quad (4)$$

The time-development of any scalar function $F(x, p)$ of the phase-space co-ordinates is then determined by the Poisson brackets

$$\frac{dF}{dt} = \{F, H\} = \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial x^i}. \quad (5)$$

In particular the Hamilton equations themselves read

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \quad (6)$$

A third formulation of the classical theory is provided by Hamilton's principal function¹ $S(x, t)$, which is the solution of the partial differential equation

$$\frac{\partial S}{\partial t} = -H(x, p = \nabla S). \quad (7)$$

For the case at hand this Hamilton-Jacobi equation takes the form

$$\frac{\partial S}{\partial t} = -\frac{1}{2} g^{ij} \nabla_i S \nabla_j S. \quad (8)$$

Particular solutions S are provided by the action for classical paths $x^i(\tau)$ obeying the Euler-Lagrange equation (2), starting at time $\tau = 0$ at an initial point $x^i(0)$, and reaching the point $x^i(t) = x^i$ at time $\tau = t$:

$$S(x, t) = \int_0^t d\tau L(x, \dot{x}) \Big|_{x^i(\tau)}. \quad (9)$$

An example of the class of theories of this type is that of a particle moving on the surface of the unit sphere, S^2 . A convenient co-ordinate system is provided by the polar angles (θ, φ) , in terms of which

$$L(\theta, \varphi) = \frac{1}{2} \left(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 \right), \quad (10)$$

for a particle of unit mass. The corresponding hamiltonian is

$$H = \frac{1}{2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right) = \frac{\mathbf{J}^2}{2}, \quad (11)$$

¹The terminology follows ref.[1].

with the momenta and velocities related by

$$p_\theta = \dot{\theta}, \quad p_\varphi = \sin^2 \theta \dot{\varphi}. \quad (12)$$

The second equality (11) relates the hamiltonian to the Casimir invariant of angular momentum, the components of which are constants of motion given by

$$J_x = -\sin \varphi p_\theta - \cos \varphi \operatorname{ctg} \theta p_\varphi, \quad J_y = \cos \varphi p_\theta - \sin \varphi \operatorname{ctg} \theta p_\varphi, \quad J_z = p_\varphi. \quad (13)$$

The geodesics on the sphere are the great circles; they can be parametrized by

$$\cos \theta(\tau) = \sin \alpha \sin \omega(\tau - \tau_*), \quad \operatorname{tg}(\varphi(\tau) - \varphi_*) = \cos \alpha \operatorname{tg} \omega(\tau - \tau_*), \quad (14)$$

where α is a constant, and τ_* and φ_* are the time and longitude at which the orbit crosses the equator: $\theta_* = \pi/2$. On these orbits the angular frequency is related to the total angular momentum by

$$\omega^2 = 2H = \mathbf{J}^2, \quad (15)$$

Observe that, for an orbit reaching the point with co-ordinates (θ, φ) at time $\tau_* + t$, the following relations hold:

$$\begin{aligned} \cos \omega &= \sin \theta \cos(\varphi - \varphi_*), & \sin \omega t &= \sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}, \\ \sin \alpha &= \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}}. \end{aligned} \quad (16)$$

The last equation implicitly describes the orbit $\theta(\varphi)$, defining a great circle which cuts the equator at $\theta = \theta_* = \pi/2$ and $\varphi = \varphi_*$, at an angle α defined by the direction of the angular momentum:

$$\frac{J_z}{\sqrt{\mathbf{J}^2}} = \cos \alpha, \quad \frac{J_\perp}{\sqrt{\mathbf{J}^2}} = \sin \alpha, \quad J_\perp = \sqrt{J_x^2 + J_y^2}. \quad (17)$$

The Hamilton-Jacobi equation for this system reads

$$\frac{\partial S}{\partial t} = -\frac{1}{2} \left[\left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right]. \quad (18)$$

The solution corresponding to the orbit (14) is

$$S(\theta, \varphi, t) = \frac{1}{2t} \arccos^2 [\sin \theta \cos(\varphi - \varphi_*)], \quad (19)$$

which satisfies the equations

$$\begin{aligned} \frac{\partial S}{\partial \theta} &= p_\theta = -\frac{\omega \cos \theta \cos(\varphi - \varphi_*)}{\sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}}, \\ \frac{\partial S}{\partial \varphi} &= p_\varphi = \frac{\omega \sin \theta \sin(\varphi - \varphi_*)}{\sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}}, \\ \frac{\partial S}{\partial t} &= -H = -\frac{\omega^2}{2}. \end{aligned} \quad (20)$$

In this approach, the expressions on the right-hand side are obtained by *defining* ω via the last expression, in agreement with (16). The same principle of energy conservation/time-translation invariance implies that S does not depend on τ_* .

2 Fluids

The Hamilton-Jacobi equation (8) can itself be obtained in a straightforward way from a variational principle: introduce a Lagrange multiplier field $\rho(x)$ and construct the action functional

$$A(\rho, S) = \int dt \int d^n x \sqrt{g} \rho \left(\frac{\partial S}{\partial t} + \frac{1}{2} g^{ij} \nabla_i S \nabla_j S \right). \quad (21)$$

The square root of the (time-independent) background metric has been included to make the integration measure invariant under reparametrizations. Of course, we could absorb it in the definition of Lagrange multiplier field, but then ρ would transform as a density rather than as scalar.

The Hamilton-Jacobi equation follows by requiring the action to be stationary w.r.t. variations of ρ :

$$\frac{1}{\sqrt{g}} \frac{\delta A}{\delta \rho} = \frac{\partial S}{\partial t} + \frac{1}{2} g^{ij} \nabla_i S \nabla_j S = 0. \quad (22)$$

On the other hand, the stationarity of $A(\rho, S)$ w.r.t. S gives

$$-\frac{1}{\sqrt{g}} \frac{\delta A}{\delta S} = \frac{\partial \rho}{\partial t} + \nabla_i (g^{ij} \rho \nabla_j S) = 0. \quad (23)$$

This equation can be interpreted as the covariant equation of continuity for a fluid² with density ρ and local velocity

$$v_i = \nabla_i S \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla_i (\rho v^i) = 0. \quad (24)$$

In this interpretation the gradient of the Hamilton-Jacobi equation gives the covariant Euler equation

$$\frac{\partial v_i}{\partial t} + v^j \nabla_j v_i = 0, \quad \nabla_j v_i = \frac{\partial v_i}{\partial x^j} - \Gamma_{ji}^k v_k. \quad (25)$$

Eq.(24) states that the fluid flow is of the potential type. Indeed, in the absence of torsion the Riemann-Christoffel connection Γ_{ij}^k is symmetric and the local vorticity vanishes:

$$\nabla_i v_j - \nabla_j v_i = 0. \quad (26)$$

For the fluid flow to be incompressible, the velocity field must be divergence free:

$$\nabla \cdot v = \Delta S = 0, \quad (27)$$

where $\Delta = g^{ij} \nabla_i \nabla_j$ is the covariant laplacean on scalar functions over the space. It follows that the number of incompressible modes of flow on the manifold equals the number of zero-modes of the scalar laplacean. For example, for flow on the sphere S^2 (or any other compact Riemann surface) there is only one incompressible mode, the trivial one with $v^i = 0$ everywhere.

For a given geometry $g_{ij}(x)$, the solution of the Hamilton-Jacobi equation (8), (22) provides a special solution of the Euler equation (25); for a conservative system: $\partial S / \partial t = -H = \text{constant}$, it implies $\partial v_i / \partial t = 0$ and $v^j \nabla_j v_i = 0$. Accordingly, this solution describes geodesic flow starting from the point (θ_*, φ_*) .

To turn this into a complete solution of the fluid-dynamical equations (24), (25) it remains to solve for the density ρ . The equation of continuity takes the form

$$\frac{\partial \rho}{\partial t} + \nabla_i (\rho \nabla^i S) = 0. \quad (28)$$

It follows that a stationary flow, with ρ not explicitly depending on time t , is possible if

$$\nabla \cdot (\rho \nabla S) = 0. \quad (29)$$

²For background, see e.g. ref.[2].

In addition to the trivial solution $\rho = \rho_0 = \text{constant}$, $v = \nabla S/m = 0$, it is possible to find non-trivial solutions of equation (29) for spatially varying density ρ . As an example, we consider flow in a 2-dimensional space; in this case one can introduce a generalized stream function $T(x, t)$, dual to the fluid momentum, and write

$$\rho \nabla^i S = \frac{1}{\sqrt{g}} \varepsilon^{ij} \nabla_j T. \quad (30)$$

Then for theories of the type (8):

$$\rho = \frac{\varepsilon^{ij} \nabla_i S \nabla_j T}{\sqrt{g} (\nabla S)^2} = \frac{\varepsilon^{ij} \nabla_i S \nabla_j T}{2H \sqrt{g}}. \quad (31)$$

With H constant, the factor $2H$ in the denominator can be absorbed into the definition of $\tilde{T} = T/2H$, and hence the density is given by

$$\rho = \frac{1}{\sqrt{g}} \varepsilon^{ij} \nabla_i S \nabla_j \tilde{T} = \frac{1}{\sqrt{g}} \varepsilon^{ij} v_i \nabla_j \tilde{T}, \quad (32)$$

for the pseudo-scalar function T the gradient of which is dual to $\rho \nabla S$. Note also, that eq.(30) implies $\nabla S \cdot \nabla T = v \cdot \nabla T = 0$.

As an illustration, we again consider the unit sphere S^2 . The velocity field is given by the momenta (20) per unit mass:

$$v_\theta = -\frac{\omega \cos \theta \cos(\varphi - \varphi_*)}{\sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}}, \quad v_\varphi = \frac{\omega \sin \theta \sin(\varphi - \varphi_*)}{\sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}}. \quad (33)$$

Taking into account that on the sphere the non-vanishing components of the connection are

$$\Gamma_{\theta\varphi}^\varphi = \frac{\cos \theta}{\sin \theta}, \quad \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta, \quad (34)$$

a straightforward calculation shows that indeed

$$v_j v^j = \omega^2, \quad v^j \nabla_j v_i = 0, \quad \frac{\partial v_i}{\partial t} = 0. \quad (35)$$

The first two equations actually imply $v^j (\nabla_i v_j - \nabla_j v_i) = 0$, in agreement with the absence of local circulation. From these results it follows, that the flowlines are geodesics (great circles) given by eq.(16), and stationary.

For the gradient of the stream function T to be orthogonal to the velocity field (33), it must satisfy the linear differential equation

$$v \cdot \nabla T = 0 \quad \Leftrightarrow \quad \text{tg}(\varphi - \varphi_*) \nabla_\varphi T = \sin \theta \cos \theta \nabla_\theta T. \quad (36)$$

The general solution can be obtained by separation of variables, and is a function of the single variable: $T(\theta, \varphi) = f(y)$, with $y = \text{tg} \theta \sin(\varphi - \varphi_*) = \text{ctg} \alpha$. For such a scalar field

$$\nabla_\theta T = \frac{\sin(\varphi - \varphi_*)}{\cos^2 \theta} f'(y)|_{y=\text{ctg} \alpha}, \quad \nabla_\varphi T = \text{tg} \theta \cos(\varphi - \varphi_*) f'(y)|_{y=\text{ctg} \alpha}. \quad (37)$$

The corresponding density ρ is then

$$\rho(\theta, \varphi) = \frac{\bar{\rho}(\alpha)}{\cos \theta} = -\frac{1}{\omega \sin \alpha \cos \theta} f'(y)|_{y=\text{ctg} \alpha}. \quad (38)$$

The simplest, most regular solution is obtained for $\bar{\rho}(\alpha) = \rho_* \sin \alpha$:

$$\rho(\theta, \varphi) = \frac{\rho_* \sin \alpha}{\cos \theta} = \frac{\rho_*}{\sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}}. \quad (39)$$

This solution corresponds to

$$T(\theta, \varphi) = \omega \rho_* \alpha(\theta, \varphi) \quad \Leftrightarrow \quad f(y) = \omega \rho_* \operatorname{arctg} y. \quad (40)$$

Observe, that in this case T , like α , is an angular variable; indeed, α increases by $2\pi n$ on any loop winding around the point $(\theta = \pi/2; \varphi = \varphi_*)$ n times.

The solution (39) possesses singular points at $\theta = \pi/2$, $\varphi = \varphi_* + n\pi$, corresponding to a source for $n = 0$, and a sink for $n = 1$. This can be established from the expression for $\nabla \cdot v$:

$$\nabla \cdot v = \frac{\omega \sin \theta \cos(\varphi - \varphi_*)}{\sqrt{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}}, \quad (41)$$

which becomes $(+\infty, -\infty)$ at the singular points. However, a more elegant way to establish the result, is to make use of the stream function (40) and consider the flux integral

$$\Phi(\Gamma) = \oint_{\Gamma} dl \rho v_n, \quad (42)$$

representing the total flow of material across the closed curve Γ per unit of time. Consider a contour Γ winding once around the singularity at $(\theta = \pi/2; \varphi = \varphi_*)$; on such a curve α increases from 0 to 2π . Then

$$\Phi(\Gamma) = \oint_{\Gamma} \sqrt{g} \varepsilon_{ij} \rho v^i dx^j = \oint_{\Gamma} \nabla_i T dx^i = 2\pi \omega \rho_*. \quad (43)$$

This represents the total flow of matter from the hemisphere centered on the source at $(\theta = \pi/2; \varphi = \varphi_*)$ to the hemisphere centered on its antipodal point, the sink at $(\theta = \pi/2; \varphi = \varphi_* + \pi)$.

3 Vortices

The dual relationship between the velocity potential S and the stream function T suggests to study the dynamics of a fluid for which T is the velocity potential:

$$v_i = \frac{1}{\rho_*} \nabla_i T. \quad (44)$$

The constant ρ_* has been included for dimensional reasons. Like before, this velocity field is stationary: $\partial v_i / \partial t = 0$, but it is not geodesic. Indeed, the velocity field describes motion under the influence of an external potential; specifically:

$$v \cdot \nabla v_i = \frac{1}{2} \nabla_i v^2 = \frac{1}{2\rho_*^2} \nabla_i (\nabla T)^2 = \frac{1}{2\rho_*^2} \nabla_i (\rho \nabla S)^2. \quad (45)$$

Here $\rho(x)$ and $S(x)$ denote the previously defined functions mapping the manifold to the real numbers—e.g. (19) and (39) for fluid motion on a sphere—irrespective of their physical interpretation. Now again, as $(\nabla S)^2 = 2H = \omega^2 = \text{constant}$, it follows that

$$v \cdot \nabla v_i = \frac{\omega^2}{2\rho_*^2} \nabla_i \rho^2 \equiv -\nabla_i h. \quad (46)$$

Combining eqs.(45) and (46):

$$\frac{1}{2} v^2 = -(h - h_0) = \frac{\omega^2 \rho^2}{2\rho_*^2}, \quad (47)$$

where h represents the external potential. Because of the potential nature of the flow, eq. (44), the local vorticity again vanishes: $\nabla_i v_j - \nabla_j v_i = 0$, but as eq.(43) shows, this is not necessarily true globally. Indeed, in singular points of the original geodesic fluid flow (with sources/sinks), the dual flow generally has vortices/anti-vortices.

Continuing our example from the previous sections, we can illustrate these results in terms of flow on the unit sphere, for which $T/\rho_* = \omega\alpha$ and $v_i = \omega\nabla_i\alpha$:

$$v_\theta = -\frac{\omega \sin(\varphi - \varphi_*)}{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}, \quad v_\varphi = -\frac{\omega \sin \theta \cos \theta \cos(\varphi - \varphi_*)}{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}. \quad (48)$$

It follows, as expected, that

$$v^2 = \omega^2 (\nabla\alpha)^2 = \frac{\omega^2 \rho^2}{\rho_*^2} = \frac{\omega^2}{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)}. \quad (49)$$

A further remarkable property, is that the dual flow is divergence free:

$$\nabla \cdot v = 0 \quad \Leftrightarrow \quad \Delta\alpha = 0, \quad (50)$$

where-ever v is well-defined; obviously, the result can only be true because of the two singular points ($\theta = \pi/2; \varphi = \varphi_*$) and ($\theta = \pi/2; \varphi = \varphi_* + \pi$), where v_i and its divergence are not well-defined, i.e. topologically the velocity field is defined on a cylinder, rather than a sphere. These two points are centers of vorticity, as follows directly from eq.(43), which in the present context can be rewritten as

$$\oint_\Gamma v_i dx^i = 2\pi\omega, \quad (51)$$

for any closed curve Γ winding once around the singular point ($\pi/2, \varphi_*$); as this curve also winds once around the other singular point in the opposite direction, they clearly define a pair of vortices of equal but opposite magnitude.

As the flow is divergence free, it follows that in this case there can be non-trivial incompressible and stationary flow modes: for constant density ρ_1 one has

$$\frac{\partial \rho_1}{\partial t} = 0, \quad \nabla \rho_1 = 0 \quad \Rightarrow \quad \nabla \cdot (\rho_1 v) = 0, \quad (52)$$

and the equation of continuity is satisfied.

The nature of the flow lines defined by eq.(48) is clear: they are parallel circles of equidistant points around the centers of vorticity. On these circles the velocity is constant in magnitude, implying by (49) that $\sin \theta \cos(\varphi - \varphi_*) \equiv \cos \beta = \text{constant}$. For example, for $\varphi_* = 0$ we get $x = \cos \beta = \text{constant}$; the flow line then is the circle where this plane of constant x cuts the unit sphere. On these flow lines

$$v_\theta = -\omega_1 \sin(\varphi - \varphi_*), \quad v_\varphi = -\omega_1 \cos \beta \cos \theta, \quad (53)$$

with

$$\omega_1 = \frac{v^2}{\omega} = \frac{\omega}{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)} = \frac{\omega}{\sin^2 \beta}. \quad (54)$$

4 The dual particle model

Having clarified the nature of the (incompressible) flow described by the dual velocity potential T/ρ_* , we now reconstruct the corresponding particle-mechanics model for which T/ρ_* is Hamilton's principal function. From eqs.(47), (49) we observe that the hamiltonian is of the form $H_1 = K + h$, with for the specific case at hand a kinetic-energy term:

$$K = \frac{1}{2} g^{ij} p_i p_j \rightarrow \frac{1}{2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right), \quad (55)$$

and the potential (normalized for later convenience such that $2H = \omega\omega_1$):

$$h(\theta, \varphi) = h_0 - \frac{\omega^2 \rho^2}{2\rho_*^2} \rightarrow \frac{\omega\omega_1}{2} \left(1 - \frac{\omega/\omega_1}{1 - \sin^2 \theta \cos^2(\varphi - \varphi_*)} \right). \quad (56)$$

The corresponding lagrangean $L_1 = K - h$ produces the Euler-Lagrange equations

$$\begin{aligned} \dot{p}_\theta &= \ddot{\theta} = \sin \theta \cos \theta \dot{\varphi}^2 + \frac{\omega^2 \sin \theta \cos \theta \cos^2(\varphi - \varphi_*)}{(1 - \sin^2 \theta \cos^2(\varphi - \varphi_*))^2}, \\ \dot{p}_\varphi &= \frac{d}{dt} (\sin^2 \theta \dot{\varphi}) = -\frac{\omega^2 \sin^2 \theta \sin(\varphi - \varphi_*) \cos(\varphi - \varphi_*)}{(1 - \sin^2 \theta \cos^2(\varphi - \varphi_*))^2}. \end{aligned} \quad (57)$$

These equations have solutions

$$\cos \theta = \sin \beta \sin \omega_1 t, \quad \text{tg}(\varphi - \varphi_*) = \text{tg} \beta \cos \omega_1 t, \quad (58)$$

with β a constant, implying the relation

$$\sin \theta \cos(\varphi - \varphi_*) = \cos \beta. \quad (59)$$

Solving for the velocity (and taking into account the unit mass)

$$p_\theta = v^\theta = -\omega_1 \sin(\varphi - \varphi_*), \quad p_\varphi = \sin^2 \theta v^\varphi = -\omega_1 \cos \beta \cos \theta, \quad (60)$$

in agreement with (53). From these results we can compute Hamilton's principal function

$$S_1(\theta, \varphi, t) = \int_0^t d\tau L_1[\theta(\tau), \varphi(\tau)] = \frac{1}{2t} \text{arcctg}^2(\text{tg} \theta \sin(\varphi - \varphi_*)). \quad (61)$$

This function indeed satisfies the Hamilton-Jacobi equations

$$\frac{\partial S_1}{\partial \theta} = p_\theta, \quad \frac{\partial S_1}{\partial \varphi} = p_\varphi, \quad (62)$$

with (p_θ, p_φ) as given by eq.(60), and

$$\frac{\partial S_1}{\partial t} = -\frac{\omega\omega_1}{2} = -\frac{1}{2} \left[\left(\frac{\partial S_1}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left(\frac{\partial S_1}{\partial \varphi} \right)^2 \right] - h(\theta, \varphi). \quad (63)$$

Using the relation $\text{ctg} \alpha = \text{tg} \theta \sin(\varphi - \varphi_*) = \text{ctg} \omega_1 t$, the equations (62) can be recast in the form

$$p_i = \omega \nabla_i \alpha = \frac{1}{\rho_*} \nabla_i T. \quad (64)$$

Hence T/ρ_* can indeed be identified with Hamilton's principal function of this system.

Repeating the arguments of sect. 2, the action (21) for the Hamilton-Jacobi theory is now generalized to:

$$A(\rho, S_1; h) = \int dt \int d^n x \sqrt{g} \rho \left(\frac{\partial S_1}{\partial t} + \frac{1}{2} g^{ij} \nabla_i S_1 \nabla_j S_1 + h \right). \quad (65)$$

Reinterpretation of S_1 as a velocity potential for fluid flow: $v = \nabla S_1$, leads back directly to the inhomogeneous Euler equation

$$\frac{\partial v_i}{\partial t} + v \cdot \nabla v_i = -\nabla_i h, \quad (66)$$

which for stationary flow becomes eq.(46). Variation of this action w.r.t. S_1 gives the equation of continuity for ρ , as before; note that in this action h plays the role of an external source for the density ρ .

5 Quantum theory

The quantum theory of a particle on a curved manifold is well-established. For the wave function to be well-defined and single-valued, the momenta must satisfy the Bohr-Sommerfeld quantization conditions

$$\oint_{\Gamma} p_i dx^i = 2\pi n \hbar, \quad (67)$$

for any closed classical orbit Γ . For the free particle of unit mass on the unit sphere the left-hand side is

$$\int_0^T v^2 d\tau = \omega^2 T = 2\pi\omega, \quad (68)$$

where $T = 2\pi/\omega$ is the period of the orbit. Hence the quantization rule amounts to quantization of the rotation frequency (the angular momentum): $\omega = n\hbar$.

For the dual model, the same quantity takes the value

$$\oint_{\Gamma} v_i dx^i = \int_0^{T_1} v^2 d\tau = \frac{\omega^2 T_1}{\sin^2 \beta} = \omega \omega_1 T_1 = 2\pi\omega, \quad (69)$$

and again $\omega = n\hbar$. As the quantization conditions in the two dual models are the same, the duality can be preserved in the quantum theory.

If this is to be true also in the fluid interpretation, the quantization conditions must be respected at that level as well. Now the first quantization condition for the integral (68) is interpreted in the fluid dynamical context as a quantization of the fluid momentum, cf. eq.(33). The second quantization condition (69) has a twofold interpretation: first, according to eqs.(42), (43) it quantizes the strength of the fluid sources and sinks in the model of free geodesic flow; the agreement between the two quantization conditions is then obvious: in order for the strength of the source/sink to satisfy a quantization condition, the amount of fluid transferred from one to the other must be quantized as well.

In the context of the dual model however, the condition imposes the quantization of vorticity in the quantum fluid [3]. In the more general context of quantum models of fluids in geodesic flow on a compact two-dimensional surface and their duals described by the stream functions, this observation shows that duality at the quantum level requires the quantization of sources in one model to be directly related to the quantization of vorticity in the dual one. This situation closely parallels the relation between the quantization of monopole charge [4] and the quantization of the magnetic flux of fluxlines [5] in three dimensions.

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